SURFACE WAVES IN AN ELASTIC MEDIUM IN THE PRESENCE OF A MAGNETIC FIELD

V. A. Feoktistov

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 7, No. 5, pp. 85–89, 1966

Waves in a perfectly conducting elastic medium are considered in two cases: 1) at the free surface of a medium occupying an infinite half-space and located in a homogeneous constant magnetic field; 2) at the interface of two media one of which is located in a magnetic field. At a certain relative velocity instability develops. A similar problem was investigated in [1] for zero magnetic field and a critical velocity was obtained. This paper examines the instability due to vibrations propagating at right angles to the magnetic field.

The equation describing the propagation of small deformations of a perfectly conducting elastic body in a homogeneous constant magnetic field H has the form

$$\rho \ddot{u}_i = \frac{\partial \Pi_{ik}}{\partial x_k}, \tag{1}$$

$$\Pi_{ik} = \sigma_{ik} + \frac{\mu}{4\pi}\,[(H^2\delta_{ik} - 2H_iH_k)\operatorname{div}\mathbf{u} + (\mathbf{H}\nabla)(H_iu_k + H_ku_i - \delta_{ik}(\mathbf{Hu}))], \tag{2}$$

$$\sigma_{ik} = G\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) + \lambda\delta_{ik}\operatorname{div}\mathbf{u},$$

$$\mathbf{H} = \{H_x, H_y, H_z\},$$

$$H_i = \text{const} \quad (i = x, y, z). \tag{3}$$

Here, ρ is the density of the medium, u the displacement vector, G, λ Lamé coefficients, and μ the permeability of the medium.

In the case of plane deformation in a transverse magnetic field, when $u_y = 0$, $\mathbf{H} = \{0, H, 0\}$ and the derivatives with respect to y are equal to zero, the wave described by Eq. (1) decomposes into two independently propagating parts, one of which describes the expansion waves, and the other the shear waves [2]. A similar decomposition is observed when the finite, but small conductivity of the medium is taken into account [3]. However, if the magnetic field lies in the plane of the displacement vector, the expansion and shear waves are interrelated. For the two nonzero components of the vector u we have the equations

$$(2G + \lambda)\frac{\partial^2 u_x}{\partial x^2} + G\frac{\partial^2 u_x}{\partial z^2} + (G + \lambda)\frac{\partial^2 u_z}{\partial x\,\partial z} - \rho\ddot{u}_x = 0,$$

$$\left(2G + \lambda + \frac{\mu H^2}{4\pi}\right)\frac{\partial^2 u_z}{\partial z^2} + \left(G + \frac{\mu H^2}{4\pi}\right)\frac{\partial^2 u_z}{\partial x^2} + (G + \lambda)\frac{\partial^2 u_x}{\partial x\,\partial z} - \rho\ddot{u}_z = 0. \tag{4}$$

In this case

$$\mathbf{H} = \{H, 0, 0\}, \qquad u_y = 0, \qquad \frac{\partial}{\partial y} = 0.$$

Starting from these equations we will determine the frequency of the surface magnetoelastic waves propagating along the magnetic field.

Let the elastic, perfectly conducting medium occupy the infinite halfspace $z < 0$. The infinite plane bounding this medium on one side is taken as the xy plane. The magnetic field is directed along the x axis and is nonzero only in the half space occupied by the medium. The solution of Eqs. (4) is represented in the form $\exp[i(\omega t - kx) + \gamma z)]$. The condition of compatibility of the two homogeneous equations, i. e., zero determinant of the system, gives a biquadratic equation for

$$a\gamma^4 + b\gamma^2 + c = 0, \qquad a = \left(2G + \lambda + \frac{\mu H^2}{4\pi}\right),$$

$$b = \frac{k^2(G+\lambda)^2}{G} - \frac{[(2G+\lambda)k^2 - \rho\omega^2]\,a}{G} + \left[\rho\omega^2 - k^2\left(G + \frac{\mu H^2}{4\pi}\right)\right],$$

$$c = -\frac{1}{G}\left[\rho\omega^2 - k^2\left(G + \frac{\mu H^2}{4\pi}\right)\right][(2G+\lambda)k^2 - \rho\omega^2],$$

whose solution is

$$\gamma_1 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)^{1/2},$$

$$\gamma_2 = \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)^{1/2} \qquad (\operatorname{Re}\gamma_1 > 0)\ (\operatorname{Re}\gamma_2 > 0).$$

Two other solutions correspond to an infinite increase in the deformations in the direction into the medium. When $H = 0$

$$\gamma_1 = \sqrt{k^2 - \rho\omega^2/(2G+\lambda)}, \quad \gamma_2 = \sqrt{k^2 - \rho\omega^2/G};$$

in this case γ_1 and γ_2 determine the dependence on z for the longitudinal and transverse waves, respectively, [4]. Thus, the solution of system (4) has the form

$$u_x = (Ae^{\gamma_1 z} + Be^{\gamma_2 z})\,e^{i(\omega t - kx)} \qquad (A, B = \text{const}),$$

$$u_z = \left(\frac{Aik\gamma_1(G+\lambda)e^{\gamma_1 z}}{\rho\omega^2 - k^2(G + {}^1\!/_4\mu H^2/\pi) + a\gamma_1^2} + \frac{Bik\gamma_2(G+\lambda)e^{\gamma_2 z}}{\rho\omega^2 - k^2(G + {}^1\!/_4\mu H^2/\pi) + a\gamma_2^2}\right)e^{i(\omega t - kx)}.$$

At the free surface the boundary condition

$$\left(\sigma_{ik}^\circ + \frac{\mu}{4\pi}\left[H_iH_k - \frac{1}{2}\delta_{ik}H^2\right] + \Pi_{ik}\right)n_k = 0 \tag{5}$$

is satisfied. Here, n is the unit vector along the normal to the perturbed surface of the medium, σ_{ik}° is the stress tensor in the unperturbed state, which has only one component $\sigma_{zz}^\circ = 1/8(\mu H^2/\pi)$.

Substituting (2) and (3) into (5) and taking into account that $n_k = \delta_{kz} - \nabla_k u_z$, we write the boundary condition in the linear approximation as follows:

$$\left(G+\frac{\mu H^2}{8\pi}\right)\frac{\partial u_z}{\partial x}+G\,\frac{\partial u_x}{\partial z}=0,$$

$$\left(2G+\lambda+\frac{\mu H^2}{4\pi}\right)\frac{\partial u_z}{\partial z}+\lambda\,\frac{\partial u_x}{\partial x}=0. \qquad (6)$$

Equating the determinant of system (6) to zero, we obtain an equation for the dependence of the frequency ω on k:

$$\gamma_1'\left[\frac{1+\nu}{1-2\sigma}+(\xi^2-1-2\nu)+\gamma_1'^2a'\right]\times$$
$$\times\left[a'\gamma_2'^2\left(\frac{1-2\sigma}{2\sigma}\right)-(\xi^2-1-2\nu)\right]=$$
$$=\gamma_2'\left[\frac{1+\nu}{1-2\sigma}+(\xi^2-1-2\nu)+\gamma_2'^2a'\right]\times$$
$$\times\left[a'\gamma_1'^2\left(\frac{1-2\sigma}{2\sigma}\right)-(\xi^2-1-2\nu)\right],$$
$$\gamma_1'=\frac{\gamma}{k},\quad \gamma_2'=\frac{\gamma_2}{k},\quad \xi^2=\frac{\rho\omega^2}{Gk^2},$$
$$\nu=\frac{\mu H^2}{8\pi G},\qquad a'=\frac{2(1-\sigma)}{(1-2\sigma)}+2\nu,$$
$$b'=\frac{1}{(1-2\sigma)^2}-\left[\frac{2(1-\sigma)}{(1-2\sigma)}-\xi^2\right]\times$$
$$\times\left[\frac{2(1-\sigma)}{(1-2\sigma)}+2\nu\right]+(\xi^2-1-2\nu),$$
$$c'=\left[\xi^2-\frac{2(1-\sigma)}{(1-2\sigma)}\right][\xi^2-1-2\nu]. \qquad (7)$$

Here, σ is Poisson's ratio. Equation (7) gives ξ as a function of the Poisson's ratio σ and the dimensionless parameter ν.

At $\nu = 0$ Eq. (7) goes over into the equation for the Rayleigh surface wave velocity. We will determine the roots of expression (7) at $\sigma = 0.25$, $\nu = 1$. By means of numerical calculation it is possible to convince oneself that in the region of real values Eq. (7) has two roots, which correspond to vibrations decaying at infinity. The first root makes the expression $(b'^2 - 4a'c')^{1/2}$ vanish and is equal to $\xi_1 = 0.618$. In the region of values of ξ satisfying the condition $(b'^2 - 4a'c') < 0$ expression (7) may be conveniently rewritten in the form

$$\mathrm{Im}\left\{\gamma_1'\left[\frac{(1+\nu)}{(1-2\sigma)}+(\xi^2-1-2\nu)+a'\gamma_1'^2\right]\times\right.$$
$$\left.\times\left[a'\gamma_2'^2\left(\frac{1-2\sigma}{2\sigma}\right)-(\xi^2-1-2\nu)\right]\right\}=0.$$

This equation is satisfied at $\xi_2 = 1.53$. At values of $\xi > \xi_2$ the roots of Eq. (7) lead to purely imaginary solutions for γ_2' and, consequently, the vibrations are sustained as $z \to -\infty$. Thus, along the magnetic field two waves may be propagated with velocities

$$\omega_1/k=\xi_1\sqrt{G/\rho},\quad \omega_2/k=\xi_2\sqrt{G/\rho}.$$

In this case the Rayleigh wave velocity (if $\sigma = 0.25$, then $\xi = 0.916$ for Rayleigh waves) lies in the interval between ω_1/k and ω_2/k.

We will consider two perfectly conducting elastic media with a plane interface, which is taken as the xy plane. As before, the magnetic field is nonzero only in the half-space $z < 0$, but is now directed along the y axis. Medium 2, which occupies the upper half-space, moves relative to medium 1 at velocity v along the x axis. We will consider an elastic wave at the interface between the two media, when the derivatives with respect to y are equal to zero and $u_y = 0$. With these assumptions the solution of Eq. (1) in region 1 is conveniently found in the form $u = \mathrm{grad}\,\varphi + \mathrm{rot}(j\psi)$,

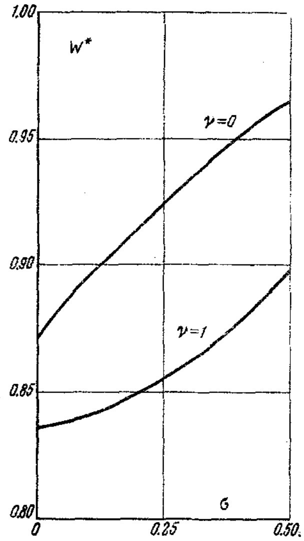

Fig. 1

where the first term describes the expansion waves and the second the shear waves. For the potentials φ and ψ we have [2]

$$\rho_1\varphi^{\cdot\cdot}-(2G_1+\lambda_1+{}^1/_4\mu H^2/\pi)\,\Delta\varphi=0,$$
$$\rho_1\psi^{\cdot\cdot}-G_1\Delta\psi=0\quad\left(\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right). \qquad (8)$$

Equations (8) make it possible to obtain the dispersion equation for the waves at the interface and to study the instability associated with these waves. The instability of relative motion of two elastic media in the presence of a magnetic field was previously examined in [5].

If the potentials φ, ψ are found in the form

$$\varphi=A'e^{ik(w_1t-x)+\alpha_1kz},\qquad \psi=B'e^{ik(w_1t-x)+\beta_1kz},$$

then from (8) we obtain

$$\alpha_1^2=1-\frac{w_1^2\rho_1}{(2G_1+\lambda_1+{}^1/_4\mu H^2/\pi)}$$
$$\beta_1^2=1-\frac{w_1^2\rho_1}{G_1}.$$

In order for the vibrations to damp in the direction into medium 1, it is necessary that Re $\alpha_1 > 0$, Re $\beta_1 > 0$. Let a weak pressure perturbation of the form $p = p_0\exp[ik(w_1t - x)]$ develop at the interface. Then the constants A', B' are found from the following boundary condition:

$$\left(\sigma_{ik}^{\circ}+\frac{\mu}{4\pi}\left[H_iH_k-\frac{\delta_{ik}}{2}H^2\right]+\Pi_{ik}\right)n_k=-pn_i. \qquad (9)$$

Substituting (2), (3) in (9) and taking into account that $n_k = \delta_{kz} - \nabla_k u_z$, we write the boundary condition in the linear approximation as follows:

$$2G_1\frac{\partial u_z}{\partial z}+\left(\lambda_1+\frac{\mu H^2}{4\pi}\right)\mathrm{div}\,u=-p,$$

$$\left(G_1 + \frac{\mu H^2}{8\pi}\right)\frac{\partial u_z}{\partial x} + G_1\frac{\partial u_x}{\partial z} = \text{[sic]}.$$

After simple calculations we can obtain an expression for the displacement of points on the surface of medium 1 along the z axis:

$$u_z^{(1)} = \frac{p}{kG_1} f_1(w_1),$$

$$\left(f_1(w_1) = -\frac{w_1^2\rho_1}{G_1}\left[\alpha_1\right]\left[2\left(2 + {}^1/_8\mu H^2/\pi G_1\right)\alpha_1\beta_1 - \right.\right.$$

$$\left.\left. -(1+\beta_1^2)\left[(1+\beta_1^2) + {}^1/_8\mu H^2/\pi G_1\right]\right]^{-1}\right). \qquad (10)$$

When H = 0 formula (10) coincides with the corresponding expression of [1]. Equating the denominator of (10) to zero, we obtain an equation for the frequency of the surface waves propagating on the free surface at right angles to the magnetic field. For fixed values of the Poisson's ration σ and the dimensionless parameter ν this equation has only one root w_1^* satisfying the condition $\operatorname{Re}\alpha_1 > 0$, $\operatorname{Re}\beta_1 > 0$. Figure 1 shows the quantity $W^* = w_1^*/(G_1/\rho)^{1/2}$ as a function of σ at $\nu = 1$. For comparison the same figure includes the analogous curve for $\nu = 0$, which determines the velocity w_2^* of the Rayleigh surface waves as a function of Poisson's ratio.

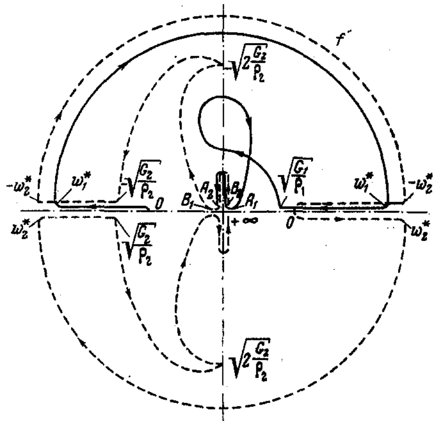

Fig. 2

The displacement $u_z^{(2)}$ for medium 2 is obtained from Eq. (10) in which the magnetic field must be set equal to zero and w_1 replaced by $w_2 = w_1 - v$. Thus,

$$u_z^{(2)} = \frac{p}{kG_2} f_2(w_2),$$

$$f_2(w_2) = -\frac{w_2^2\rho_2}{G_2}\,\frac{\alpha_2}{4\alpha_2\beta_2 - (1+\beta_2^2)^2},$$

$$\alpha_2^2 = 1 - w_2^2\rho_2/(2G_2 + \lambda_2),$$

$$\beta_2^2 = 1 - w_2^2\rho_2/G_2.$$

In this case $\operatorname{Re}\alpha_2 < 0$, $\operatorname{Re}\beta_2 < 0$, since medium 2 occupies the half-space $z > 0$.

From the condition of equality of the displacements of surface points of media 1 and 2 we find an equation for the relation between w_1 and k:

$$G_1^{-1}f_1(w_1) = G_2^{-1}f_2(w_2),$$

$$w_2 = w_1 - v. \qquad (11)$$

If an incompressible perfect fluid flows over the elastic magnetized medium, then instead of (11) we have the equation

$$G_1^{-1}f_1(w_1) = -w_2^{-2}\rho_2^{-1}.$$

When the complex velocity, determined in solving Eqs. (11), has a negative imaginary part, the vibrations will increase with time, and the state of the system will be unstable.

Let $w_1 = a_1 - i\varepsilon$, where $0 < a_1 < +\infty$; then the function f_1 on the complex plane f gives a certain curve, whose form is represented in Fig. 2 by a solid line. The arrow indicates the direction to be followed around the curve as a_1 increases from 0 to $+\infty$. The point A_1 corresponds to the value of the function f_1 at $a_1 = [\rho_1^{-1}(2G_1 + \lambda_1 + \mu H^2/4\pi]^{1/2}$, and the point B_1 to the value of the function at $a_1 = +\infty$. Under the transformation effected by the function f_2 a straight line parallel to the real axis and displaced through an infinitely small distance into the lower half plane goes over into the curve denoted by a dashed line in Fig. 2. The arrow indicates the direction to be followed around the curve as a_2 varies from $-\infty$ to $+\infty$; in this case when a_2 changes sign the curve passes from the upper to the lower half-plane. The form of this curve is essentially the same as that presented in [1].

The points A_2 and B_2 correspond to values of the function f_2 at $a_2 = -((2G_2 + \lambda_2)\rho_2^{-1})^{1/2}$ and $a_2 = -\infty$, respectively. The intersection of the solid and dashed lines means that Eqs. (11) admit solutions for the velocity w_1 having a negative imaginary part.

If $v = 0$, then $a_2 > 0$ (considering that $a_2 = a_1 - v$) and f_2 gives only that part of the curve located in the lower half plane.

This corresponds to Eqs. (11) having a solution only at real values of w_1. If $v \neq 0$, then, as may be seen from Fig. 2, the dashed and solid lines may intersect, this intersection occurring when:

(1) $a_1 < \sqrt{G_1/\rho_1}$, $\quad |a_2| < \sqrt{G_2/\rho_2}$,

(2) $a_1 > \sqrt{(2G_1 + \lambda_1 + {}^1/_4\mu H^2/\pi)\rho_1^{-1}}$,

$|a_2| > \sqrt{(2G_2 + \lambda_2)\rho_2^{-1}}$.

Since $a_1 + |a_2| = v$, the first condition corresponds to smaller values of the relative velocity. If the intersection occurs in the first octant, then $w_1^* < v < (w_2^* + (G_1/\rho_1)^{1/2}$; similarly, for the second octant $w_2^* < v < w_1^* + (G_2/\rho_2)^{1/2}$. At velocities $v < \min\{w_1^*, w_2^*\}$ there are only real solutions. In this case, in complete correspondence with [1], the solution is determined from a

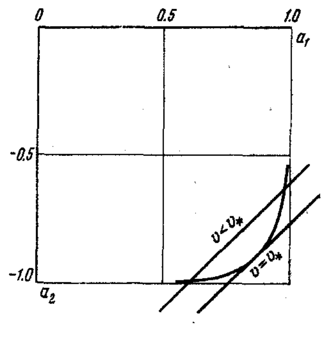

Fig. 3

graphic construction. For this purpose, starting from the first of Eqs. (11), we construct a graph of $w_1 = F(w_2)$, when $0 < a_2 | < (G_2/\rho_2)^{1/2}$, and on it plot the straight line $a_2 = a_1 - v$.

The point of intersection determines the solution (Fig. 3). At $v > v_*$ the solution of Eqs. (11) passes into the region of complex values of w_1 having a negative imaginary component. A similar argument can also be made for negative values of a_1. We determined the critical velocity v_* for two identical media at $\nu = 1$ and $\sigma = 2/7$; it was found that $v_* = 1.78(G/\rho)^{1/2}$. At $\nu = 0$, in accordance with the results of [1], the critical velocity is equal to twice the Rayleigh wave velocity, i.e., $v_* = 1.85(G/\rho)^{1/2}$. Thus, a transverse magnetic field has almost no effect on the critical velocity.

The author thanks M. I. Kiselev for his helpful suggestions.

REFERENCES

1. R. E. Lamper, "Stability of relative motion of media," Izv. VUZ. Aviatsionnaya tekhnika, no. 4, 32–39, 1959.

2. N. I. Dolbin, "Propagation of elastic surface waves in a half-space located in a magnetic field," PMTF, no. 1, 84–87, 1963.

3. Ya. S. Uflyand, "An approximate method of solving problems of magnetoelasticity for bodies of finite conductivity," PMTF, [Journal of Applied Mechanics and Technical Physics], no. 2, 155–157, 1965.

4. L. D. Landau and E. M. Lifshits, Theory of Elasticity [in Russian], Nauka, 1965.

5. S. Kaliski, "Stability of relative motion of two perfectly conducting elastic media in a magnetic field parallel to the direction of motion," Proc. of Vibration Problems, Warsaw, 1964, vol. 5, no. 2, 75–86, 1964.

15 February 1966 Moscow